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Classical $D = 2+1$ spinning string: geometrical description and current algebras

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Abstract. A theory for a classical $D = 2+1$ string with a distributed spinor field is suggested. This field is defined by real Majorana spinors which are spinors both in initial three-dimensional spacetime and in corresponding touching planes. It is shown that the dynamics of that object in terms of Poisson brackets is defined by the pair of algebras of $(\mathrm{so}(1,2)\otimes[t^{-1}, t])\oplus C_z$ type. A one-to-one correspondence between this model and the conformal-invariant model of two-dimensional scalar and spinor fields with non-trivial interaction (Thirring × Liouville model) is determined.

1. Introduction

As it now seems, the algebraic structure of hadronic physics is connected with some type of affine Lie algebras—current algebras. These objects appear in various models of strongly interacting particles (see for example De Alfaro *et al* 1973). It is widely known that the quantum string theory is not an exception here (Dolan 1984). As regards the classical string, a Virasoro algebra, and not a current algebra, is generally considered the foundation of Poisson bracket structure for the theory. In this paper we want to show another example, where the Hamiltonian structure of the classical $D = 2+1$ string with spinor degrees of freedom is defined by the pair of current algebras. The latter contain non-zero central charges. Virasoro commutators (in the sense of Lie operators) are present in the theory, but it is the consequence of current commutators which seem to be most fundamental here.

The object of our investigation—the spinor string—is described by the pair of coordinates (X_μ, Ψ_j^a), $\mu = 0, 1, 2$, $a, j = 1, 2$, where $X_\mu = X_\mu(\xi^0, \xi^1)$, the coordinates of the string in three-dimensional spacetime $M_{1,2}$ with metric $(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1)$, and $\Psi_j^a = \Psi_j^a(\xi^0, \xi^1)$, the components of spinor field, defined on the world surface $\{X_\mu\}$. Two types of indexes here—a and j—mean that (complex in general, but not Grassmann!) numbers Ψ_j^a define both spinors in initial space $M_{1,2}$ (index a) and spinors in two-dimensional planes which touch the world surface $\{X_\mu\}$ at the point $X_\mu(\xi^0, \xi^1)$. Sometimes we shall drop a certain index: the notation Ψ^a, $a = 1, 2$, means the pair of two-dimensional spinors and Ψ_j, $j = 1, 2$, means the pair of three-dimensional spinors.

The action functional for our string is analogous to superstring theory (Schwarz 1982). We use an orthonormal gauge in this work:

$$(\partial_\pm X_\mu)^2 = 0 \tag{1}$$

where

$$\partial_\pm \equiv \partial/\partial\xi_\pm \qquad \xi_\pm \equiv \xi^1 \pm \xi^0.$$

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In this gauge the action functional is

$$S = -\int \mathrm{d}\xi^0\,\mathrm{d}\xi^1\{\partial_+X_\mu\partial_-X^\mu + \mathrm{i}\varepsilon_{aB}\bar{\Psi}^a\gamma^i\partial_i\Psi^B\} \tag{2}$$

where $\gamma^0 \equiv \sigma_1$, $\gamma^1 \equiv \mathrm{i}\sigma_2$ (Pauli matrices), $\bar{\Psi}^a \equiv \Psi^{+a}\gamma^0$, $\partial_j \equiv \partial/\xi_j$, ε_{ab} is the invariant spinor for $M_{1,2}$: for some three-dimensional spinors φ, ν the function $\varepsilon_{ab}\varphi^{*a}\nu^b$ is scalar.

In this paper we shall consider the closed string:

$$X_\mu(\xi^0, 0) = X_\mu(\xi^0, \pi).$$

For spinor coordinates the following conditions take place: either

$$\Psi^a(\xi^0, 0) = \Psi^a(\xi^0, \pi) \tag{3a}$$

or

$$\Psi^a(\xi^0, 0) = -\Psi^a(\xi^0, \pi) \tag{3b}$$

by analogy with the Ramon and Neveu-Schwarz spinning string, respectively.

The action (2) leads to the following equations of motion:

$$\partial_+\partial_-X_\mu = 0 \qquad \mu = 0, 1, 2 \tag{4a}$$

$$\mathrm{i}\gamma^j\partial_j\Psi^a = 0 \qquad a = 1, 2. \tag{4b}$$

For convenience note $\Psi_1^a \equiv \Psi_+^a$, $\Psi_2^a = \Psi_-^a$. It is obvious that this notation is motivated by equations $\partial_\pm\Psi_\mp^a = 0$, which are equivalent to (4b).

We suppose that the three-dimensional spinors $\Psi_\pm$ are Majorana spinors. Because the pure imaginary representation $\Gamma^0 = \sigma_2$, $\Gamma^1 = \mathrm{i}\sigma_1$, $\Gamma^2 = \mathrm{i}\sigma_3$ for Dirac matrices Γ^μ will be used, the numbers $\psi_\pm{}^a$ are real numbers (Sherck 1979). Next, define the pair of light-like vectors:

$$J_\pm{}^\mu = \mp\tfrac{1}{2}\bar{\Psi}_\pm\Gamma^\mu\Psi_\pm \qquad \bar{\Psi}_\pm \equiv \Psi_\pm{}^+\Gamma^0.$$

With the help of the quantities $J_\pm{}^\mu$ we formulate additional conditions, the fulfilment of which is also important. These conditions are

$$\partial_\pm X_\mu J_\pm{}^\mu \neq 0. \tag{5}$$

In our point of view the model outlined above can be considered as the generalisation of a standard Nambu-Goto (NG) string in the light-like gauge. Indeed, take

$$\Psi_+^a(\xi_+) \equiv \Psi_-^a(\xi_-) \equiv \text{constant}.$$

This supposition means that ${}^\pm J_\mp{}^\mu = n^\mu$ is the constant light-like vector and we have additional conditions which are standard for NG theory (Mandelstam 1974, Barbashov and Nesterenko 1987):

$$\partial_\pm X_\mu n^\mu \neq 0. \tag{6}$$

It should be emphasised here that condition (5), unlike (6), can be considered as the restriction only for spinors $\Psi_\pm$ but not for coordinates X_μ.

The purpose of this work is, first, to investigate the geometry of the object outlined above and to deduce the corresponding system of non-linear differential equations and, second, to study string Hamiltonian dynamics as Hamiltonian dynamics of the solutions of these equations. This method is not new. The standard geometrical description of an NG string ($D = 2+1$) in terms of the two-dimensional Liouville equations $\Box\varphi+\exp\varphi=0$ is well known (Barbashov and Nesterenko 1980). As demonstrated above, our approach generalises the standard one and takes into account spinor degrees of freedom of every string point. The suggested theory leads to the current algebra. The appearance of this fundamental object on a classical level seems to be interesting.

2. Geometrical description

Now let us consider real functions $X^\mu=X^\mu(\xi^0,\xi^1)$ and $\Psi_\pm{}^a=\Psi_\pm{}^a(\xi^0,\xi^1)$ which satisfy (4) and (5). The equations of motion and additional conditions are invariant under the two-dimensional conformal group G_2, i.e. the group of transformations

$$\xi_\pm\to\tilde{\xi}_\pm=f_\pm(\xi_\pm) \tag{7}$$

where $f_\pm(\xi_\pm)$ are arbitrary regular functions with $f'_\pm\neq 0$. To fix the parametrisation (ξ^0,ξ^1) conformal freedom (7) has to be destroyed. Let us do it, demanding

$$\partial_\pm X_\nu J_\pm{}^\nu=\tfrac{1}{2}s^{\mp 2}\qquad s=\text{constant}. \tag{8}$$

Note for convenience $x_\pm{}^\mu\equiv S^{\pm 1}\partial_\pm X^\mu$, $y_\pm{}^\mu\equiv s^{\pm 1}J_\pm{}^\mu$ and introduce $z_\pm{}^\mu=2\varepsilon^{\mu\nu\lambda}y_{\pm\nu}x_{\pm\lambda}$.

We have $(x_\pm)^2=(y_\pm)^2=0$ and, because of (8), $(z_\pm)^2=-1$. This means that 'plus' vectors (x_+,y_+,z_+) and 'minus' vectors give the pair of basis in initial spacetime $M_{1,2}$. Let $B_\pm$ be matrices of its coordinates in some constant basis $\boldsymbol{\eta}_\alpha$ ($\boldsymbol{\eta}_\alpha\boldsymbol{\eta}_\beta=g_{\alpha\beta}$, $\alpha,\beta=0,1,2$)

$$B_\pm=\pm\begin{pmatrix}x_{\pm 0} & x_{\pm 1} & x_{\pm 2}\\ y_{\pm 0} & y_{\pm 1} & y_{\pm 2}\\ z_{\pm 0} & z_{\pm 1} & z_{\pm 2}\end{pmatrix}.$$

Denote by $\mathcal{K}(\xi^0,\xi^1)$ the matrix, which connects B_- and B_+:

$$B_-(\xi_-)=\mathcal{K}(\xi^0,\xi^1)B_+(\xi_+). \tag{9}$$

Because of the local isomorphism SL(2, R) and the connected components of SO(2, 1) we have

$$\mathcal{K}(\xi^0,\xi^1)=\begin{pmatrix}K_{12}^2 & K_{11}^2 & K_{12}K_{11}\\ K_{22}^2 & K_{21}^2 & K_{21}K_{22}\\ 2K_{12}K_{22} & 2K_{11}K_{21} & K_{11}K_{22}+K_{12}K_{21}\end{pmatrix}$$

where $K_{ij}=K_{ij}(\xi^0,\xi^1)$ are the elements of the real 2×2 matrix K with the unit determinant. Next, the equality (9) gives

$$\partial_-(\mathcal{K}^{-1}\partial_+\mathcal{K})=0.$$

Hence, unambiguously,

$$\partial_-(K^{-1}\partial_+K)=0. \tag{10}$$

Now we write the Gauss decomposition for the matrix K as the element SL(2, R) (Barut and Raczka 1977):

$$K=A_-^{-1}\phi A_+ \tag{11}$$

where

$$A_+=\begin{pmatrix}1 & \alpha_+\\ 0 & 1\end{pmatrix}\qquad A_-=\begin{pmatrix}1 & 0\\ \alpha_- & 1\end{pmatrix}\qquad \phi=\begin{pmatrix}\exp(-\varphi/2) & 0\\ 0 & \exp\varphi/2\end{pmatrix}.$$

It is well known that between the matrices $K \in \mathrm{SL}(2, R)$ and the set of numbers $(\varphi, \alpha_+, \alpha_-)$ a one-to-one correspondence exists, except for some group points for which

$$K_{11}(\xi^0, \xi^1) \equiv \exp\left(-\frac{\varphi(\xi^0, \xi^1)}{2}\right) = 0. \tag{12}$$

In our case this equality sets a correspondence between singular elements of SL(2, R) and points of singularities of functions φ, $\alpha_\pm$ in the (ξ^0, ξ^1) plane.

Because of (10) and (11) for regular points we have

$$-\tfrac{1}{2}\partial_+\partial_-\varphi + (\partial_-\alpha_+)(\partial_+\alpha_-)\exp(-\varphi) = 0 \tag{13b}$$

$$\partial_\mp[\partial_\pm\alpha_\mp \exp(-\varphi)] = 0. \tag{13b}$$

Denote $\partial_\pm\alpha_\mp \exp(-\varphi) \equiv \rho_\mp$ and rewrite system (13) in the form of

$$-\tfrac{1}{2}\partial_+\partial_-\varphi + \rho_+\rho_- \exp(\varphi) = 0 \tag{14a}$$

$$\partial_\pm\rho_\mp = 0 \tag{14b}$$

$$\partial_\pm\alpha_\mp = \rho_\mp \exp(\varphi). \tag{14c}$$

Equations (14) are the Lagrange–Euler equations for action:

$$S_g = \int \mathscr{L}(\xi^0, \xi^1)\, \mathrm{d}\xi^0\, \mathrm{d}\xi^1 \tag{15}$$

where

$$\mathscr{L}(\xi^0, \xi^1) = \tfrac{1}{4}(\partial_+\varphi)(\partial_-\varphi) + \rho_+\rho_- \exp(\varphi) - \rho_+\partial_-\alpha_+ - \rho_-\partial_+\alpha_-.$$

The action (15) was first used in the work of Pogrebkov and Talalov (1987) for constructing a two-dimensional field theory model. This model describes a non-trivial interaction of a scalar field φ and spinor field $\Xi = (\Xi_+, \Xi_-)^{\mathrm{T}}$ with components

$$\Xi_\pm \equiv (\rho_\pm)^{1/2} \exp(\pm 4\mathrm{i}\alpha_\pm).$$

Interaction of the fields is defined here by means of part of the Lagrangian:

$$\rho_+\rho_- \exp(\varphi) \sim (\bar{\Xi}\gamma^\mu\Xi)^2 \exp(\varphi)$$

which unites the widely known Thirring and Liouville two-dimensional field theories. It was shown there that the model possesses conformal-invariance properties.

Now let us discuss the system (14) from the view point of classical differential geometry of the world surface $\{X_\mu\}$. According to the definition of the $\mathscr{K}$ matrix and field $\varphi(\xi^0, \xi^1)$ we have

$$\partial_+X_\mu\partial_-X^\mu = -\tfrac{1}{2}\exp(-\varphi). \tag{16}$$

This means that the function $\varphi(\xi^0, \xi^1)$ defines the first quadratic form of the world surface $\{X_\mu\}$:

$$\mathrm{d}s^2 = \exp[-\varphi(\xi^0, \xi^1)][(\mathrm{d}\xi_0)^2 - (\mathrm{d}\xi_1)^2].$$

Then we write for the coefficient of the first and second (b_{ij}) quadratic forms the Peterson–Codazzi and Gauss equation (see, for example, Dubrovin *et al* (1979) and Barbashov and Nesterenro (1987)):

$$\partial_\pm(b_{11} \mp b_{12}) = 0 \tag{17a}$$

$$R_{1212} = 2\exp(-\varphi)\partial_+\partial_-\varphi = (b_{11} + b_{12})(b_{11} - b_{12}) \tag{17b}$$

where R_{1212} is the non-trivial component of the Riemann tensor for the surface $\{X_\mu\}$. Because of (14) we have

$$(b_{11} \pm b_{12})^2 = 4(\text{constant})^{\pm 1} \rho_\pm^2. \tag{18}$$

For coordinates X_μ this means (Barbashov and Nesterenko 1987)

$$(\partial_\pm^2 X_\mu)^2 = -\tfrac{1}{4}(b_{11} \pm b_{12})^2 = -(\text{constant})^{\pm 1} \rho_\pm^2.$$

The following point has to be stressed here. Assume that the second quadratic form b_{ij} of the surface satisfies the condition

$$b_{11} \pm b_{12} \neq 0.$$

This means that, in accordance with conformal freedom (7), $b_{11} \pm b_{12} = \text{constant}$ can be chosen, and we have a standard geometrical description of the string in terms of one function $\tilde{\varphi}(\xi^0, \xi^1)$ which gives the Liouville equation. If equalities $b_{11} \pm b_{12} = 0$ are permitted, there is no such conformal equivalence. In our case, functions $\rho_\pm(\xi_\pm)$ are dynamical variables; they can take zero values on some intervals, probably.

3. Cauchy problem

Our next purpose is to express the solutions of equations (4) through Cauchy data for the system (14). First we should say how the Cauchy task for (14) was solved in Pogrebkov and Talalov (1987) and Talalov (1987). In those works the non-periodic case $-\infty \leqslant \xi^1 \leqslant \infty$ was considered. According to (12), the Cauchy initial data for (14), $\varphi(\xi) \equiv \varphi(0, \xi)$, $\pi(\xi) \equiv \partial\varphi(0, \xi)/\partial\xi^0$, $\alpha_\pm(\xi) \equiv \alpha_\pm(0, \xi)$, can possess, by supposition, a finite number of singularities—a logarithmic type for $\varphi(\xi)$ and a pole type for $\pi(\xi)$ and $\alpha_\pm(\xi)$. The initial data $\rho_\pm(\xi) \equiv \rho_\pm(0, \xi)$ are always regular. Define matrices

$$U(\xi^0, \xi^1) = \begin{pmatrix} \dfrac{1}{4}\dfrac{\partial\varphi}{\partial\xi^0} & \rho + \exp\left(\dfrac{\varphi}{2}\right) \\ \rho = \exp\left(-\dfrac{\varphi}{2}\right) & -\dfrac{1}{4}\dfrac{\partial\varphi}{\partial\xi^0} \end{pmatrix}$$

$$F_\pm(\xi^0, \xi^1) = \begin{pmatrix} 0 & \exp(\pm\varphi/4) \\ -\exp(\mp\varphi/4) & 0 \end{pmatrix}$$

with $\varphi \equiv \varphi(\xi^0, \xi^1)$, $\rho_\pm \equiv \rho_\pm(\xi_\pm)$ and matrices

$$Q_\pm(\xi^0, \xi^1) = A_\pm[F_\pm[U(\xi^0, \xi^1)]]$$

where the notation $C[B]$ for gauge transformation matrix B with the help of matrix C is used: $C[B] \equiv C^{-1}BC - (\partial C^{-1}/\partial\xi^1)C$. The importance of the traceless matrices $Q_\pm(\xi^0, \xi^1)$ is determined by the following qualities. First, the elements of $Q_\pm$ are regular functions even in singular for the φ, π, $\alpha_\pm$ case and, second, the equations for $Q_\pm(\xi^0, \xi^1)$ are quite simple:

$$\partial_\pm Q_\mp(\xi^0, \xi^1) = 0. \tag{19}$$

Equalities (19) lead to $Q_\pm(\xi^0, \xi^1) = Q_\pm(0, \xi_\pm)$, where $Q_\pm(0, \xi_\pm) \equiv Q_\pm(\xi_\pm)$ are matrices, which are constructed by means of the Cauchy task initial data only. The Cauchy task is solved for (14) as follows. Obviously, the reconstruction $\rho_\pm(\xi^0, \xi^1) = \rho_\pm(\xi_\pm)$ is trivial. Let $T_\pm(\xi)$ be matrices of solutions of the auxiliary linear regular 2×2 systems:

$$T'_\pm(\xi) + Q_\pm(\xi) T_\pm(\xi) = 0. \tag{20}$$

Moreover it is supposed that

$$T_{\pm}(\xi) = I \qquad \text{for} \qquad \xi = \zeta. \tag{21}$$

In Talalov (1987) $\zeta = +\infty$ was chosen, but in our periodic case for $Q_{\pm}(\xi)$ we have to put $\zeta \in [0, \pi]$; we can now put $\zeta = 0$, for example.

It is stated that the matrix $K(\xi^0, \xi^1)$ (see (11)) is

$$K(\xi^0, \xi^1) = T_{-}(\xi_{-}) T_{+}^{-1}(\xi_{+}). \tag{22}$$

The condition (21) destroys three-parameter freedom in the choice of matrices $T_{\pm}$:

$$T_{\pm} \to \tilde{T}_{\pm} = T_{\pm} B \tag{23}$$

where $B = \text{constant}$, $\det B = 1$.

To express the coordinates X_{μ}, $\mu = 0, 1, 2$, through Cauchy data φ, π, $\alpha_{\pm}$, $\rho_{\pm}$ it is convenient to take a constant basis in $M_{1,2}$ in the form of $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}_2)$, where $\boldsymbol{\alpha} = \frac{1}{2}(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_0)$, $\boldsymbol{\beta} = \frac{1}{2}(\boldsymbol{\eta}_1 + \boldsymbol{\eta}_0)$. According to the results of Omnes (1979) and Barbashov and Nesterenko (1987) for derivatives of the local minimal world surface in orthonormal gauge and for its first quadratic form coefficients we have

$$x_{\pm} = \pm(\rho_{\pm}/f'_{\pm})[\boldsymbol{\alpha} - (f_{\pm})^2 \cdot \boldsymbol{\beta} \pm f_{\pm} \cdot \boldsymbol{\eta}_2] \tag{24}$$

$$\exp(-\varphi) = (\rho_{+}\rho_{-}/f'_{+}f'_{-}) \cdot [f_{+} + f_{-}]^2. \tag{25}$$

Functions $f_{\pm} \equiv f_{\pm}(\xi_{\pm})$ are arbitrary here. In our case $\exp(-\varphi) = K_{11}^2$ and because of (22) we have (to an accuracy of transformation (23))

$$f_{+} = -(t_{22}^{+}/t_{21}^{+}) \qquad f'_{+} = \rho_{+}/(t_{21}^{+})^2$$

$$f_{-} = t_{12}^{-}/t_{11}^{-} \qquad f_{-}^{a0} = \rho_{-}/(t_{11}^{-})^2$$

where $(t_{ij}^{\pm}) \equiv (T_{\pm})_{ij}$.

The resulting formulae are

$$\begin{aligned} x_{+} &= (t_{21}^{+})^2 \cdot \boldsymbol{\alpha} - (t_{22}^{+})^2 \cdot \boldsymbol{\beta} - t_{21}^{+} t_{22}^{+} \cdot \boldsymbol{\eta}_2 \\ x_{-} &= -(t_{11}^{-})^2 \cdot \boldsymbol{\alpha} + (t_{12}^{-})^2 \cdot \boldsymbol{\beta} + t_{11}^{-} t_{12}^{-} \cdot \boldsymbol{\eta}_2. \end{aligned} \tag{26}$$

These expressions restore the derivatives of the world surface $\{X_{\mu}\}$ through the Cauchy data of system (14). Direct verification shows that transformations (23) induce Lorentz transformations of the coordinates X_{μ}. Choose as the real Majorana spinors Ψ_{+} and Ψ_{-}

$$\Psi_{+} = \pm\frac{1}{\sqrt{s}}\begin{pmatrix} -t_{12}^{+} \\ t_{11}^{+} \end{pmatrix} \qquad \Psi_{-} = \pm\sqrt{s}\begin{pmatrix} -t_{22}^{-} \\ t_{21}^{-} \end{pmatrix}. \tag{27}$$

Because matrices (B) and $(-B)$ in (23) give the same transformation of the vectors X_{μ}, the transformations (23) are spinor here

$$\Psi_{\pm} \to \tilde{\Psi}_{\pm} = B^{-1}\Psi_{\pm}.$$

Next, we have

$$\begin{aligned} y_{+} &= (t_{11}^{+})^2 \cdot \boldsymbol{\alpha} - (t_{12}^{+})^2 \cdot \boldsymbol{\beta} - t_{11}^{+} t_{12}^{+} \cdot \boldsymbol{\eta}_2 \\ y_{-} &= -(t_{21}^{-})^2 \cdot \boldsymbol{\alpha} + (t_{22}^{-})^2 \cdot \boldsymbol{\beta} + t_{21}^{-} t_{22}^{-} \cdot \boldsymbol{\eta}_2. \end{aligned} \tag{28a}$$

The additional conditions (8) can be verified directly. In accordance with periodicity $X_\mu(\xi^1)$ and condition (3) we have to demand

$$M_+ = M_- = \pm I \tag{28b}$$

where $M_\pm$ are the monodromy matrices for system (20):

$$T_\pm(\xi+\pi) = T_\pm(\xi)M_\pm .$$

Thus, we have reconstructed the initial object $(X_\mu, \Psi^a_\pm)$ through Cauchy data for system (14). The model is quite 'supersymmetric': we can introduce a string Y_μ with derivatives of the world surface:

$$\partial_\pm Y^\mu = J_\pm{}^\mu$$

and spinors $\nu_\pm$ with the components

$$\nu_+ = \pm\frac{1}{\sqrt{s}}\begin{pmatrix} -t^+_{22} \\ t^+_{21}\end{pmatrix} \qquad \nu_- = \pm\sqrt{s}\begin{pmatrix} -t^-_{12} \\ t^-_{11}\end{pmatrix}$$

so that

$$\partial_\pm X^\mu = \mp\bar{\nu}_\pm \Gamma^\mu \nu_\pm .$$

Due to the demonstrated symmetry

$$(X^\mu, \Psi^a_\pm) \leftrightarrow (\nu^a_\pm, Y^\mu) \tag{29a}$$

representation $(28a)$ for vectors $y_\pm$, and, consequently, representation (27) for spinors $\Psi_\pm$ through the initial Cauchy data φ, π, $\alpha_\pm$, $\rho_\pm$, is single valued to an accuracy of transformation (23).

To fix the correspondence completely, we have to add some condtions on the initial variables X_μ and $\Psi^a_\pm$ at the point $\xi^0 = 0$, $\xi^1 = 0$:

$$\partial_\pm X(0,0) = -\tfrac{1}{2}s^{\mp 1}(\eta_1 \pm \eta_0)$$

$$\Psi_+(0,0) = \frac{1}{\sqrt{s}}\begin{pmatrix} 0 \\ \pm 1\end{pmatrix} \qquad \Psi_-(0,0) = \sqrt{s}\begin{pmatrix} \pm 1 \\ 0\end{pmatrix}. \tag{29b}$$

It is clear that conditions $(29b)$ can be varied.

Next, we have to demand that the 1-form $\mathrm{d}X_\mu$ and elements of the matrix $K(\xi^0, \xi^1)$, which are quite geometrical, be invariant as regards Lorentz transformations of the (ξ^0, ξ^1) plane. This means that transformations

$$\xi_\pm \to \tilde{\xi}_\pm = \lambda^{\pm 1}\xi_\pm \qquad \lambda > 0$$

lead to (see (8) and (20))

$$s \to \tilde{s} = \lambda s$$

$$T_\pm \to \tilde{T}_\pm(\tilde{\xi}_\pm) = T_\pm(\lambda^{\pm 1}\xi_\pm).$$

These formulae correspond to the assumption that objects $\psi^a = (\Psi^a_+, \Psi^a_-)^{\mathrm{T}}$ are two-dimensional spinors for every three-dimensional spinor index.

4. Poisson bracket structure

We now describe Hamiltonian dynamics of the object considered. As usual, we assume that ξ^0 is the 'time' parameter, which defines the evolution of system. A suitable Poisson bracket structure is the first of what is necessary to fulfil our intentions. Formally, let

$$\begin{aligned}&\{\pi(\xi), \varphi(\eta)\} = \delta(\xi-\eta)\\&\{\alpha_\pm(\xi), \rho_\pm(\eta)\} = \pm\tfrac{1}{4}\delta(\xi-\eta)\end{aligned} \tag{30}$$

and the rest of the possible brackets are equal to zero. In the regular case for initial data, equalities (30) can be realised by means of the widely known definition of Poisson bracket structure:

$$\{F, G\} = \int \mathrm{d}y \left(\frac{\delta F}{\delta\pi(y)}\frac{\delta G}{\delta\varphi(y)} - (F\leftrightarrow G)\right) + \ldots .$$

But, in accordance with the procedure of Gauss decomposition (11), singular solutions of system (14) must be considered as well. Formulae (30) in this case are ambiguous, because 'little variations' $\delta\varphi(\xi)$, $\delta\pi(\xi)$, $\delta\alpha_\pm(\xi)$ are not well defined values.

Beginning with (30) we have

$$\{Q_\pm(\xi) \underset{,}{\otimes} Q_\pm(\eta)\} = 2R_\pm\delta'(\xi-\eta) + [R_\pm, 1_2\otimes Q_\pm(\eta) - Q_\pm(\xi)\otimes 1_2]\delta(\xi-\eta) \tag{31a}$$

$$\{Q_+(\xi) \underset{,}{\otimes} Q_-(\eta)\} = 0 \tag{31b}$$

where $R_\pm = \mp\frac{1}{16}(2P - 1_4)$, P is the rearrangement matrix: $(P(A\otimes B) = (B\otimes A)P$, the square brackets denote the commutator of correspondent matrices, and $\otimes$ and $\underset{,}{\otimes}$ are standard tensor notation. Note that brackets (31) are a pair of second Hamiltonian structures for a non-linear Shrödinger equation (Magri 1978, Kulish and Rayman 1978). We emphasise that elements of the matrices $Q_\pm$ are always regular functions, even in singular for the $\varphi(\xi)$, $\pi(\xi)$, $\alpha_\pm(\xi)$ cases. Taking this into account, we postulate brackets (31) as fundamental ones in the string model studied. The detailed construction of the Poisson structure of system (14) with the help of (31) was made in Talalov (1987). In this work regular canonical variables of action-angle type and canonical (Noether) energy-momentum tensors were constructed. The method was first suggested in Jorjadze *et al* (1986) for investigation of the singular case of Liouville's equation solutions.

For energy H we have

$$H = \frac{1}{2}\int_0^\pi \mathrm{d}\xi(\theta_+ + \theta_-) \tag{32}$$

where

$$\theta_\pm \equiv 4\,\mathrm{tr}[Q_\pm]^2.$$

Next, make the decompositions

$$Q_\pm(\xi) = \mp\tfrac{1}{4}\mathrm{i}j^{\pm\mu}\Gamma_\mu.$$

Then, due to (31), we have

$$\{j^\pm{}_\mu(\xi), j^\pm{}_\nu(\eta)\} = \pm 2g_{\mu\nu}\delta'(\xi-\eta) + \varepsilon_{\mu\nu\lambda}j^{\pm\lambda}(\xi)\delta(\xi-\eta) \tag{33a}$$

$$\{j^+{}_\mu(\xi), j^-{}_\nu(\eta)\} = 0. \tag{33b}$$

Thus we describe the initial object—closed classical $D=2+1$ string with the distributed spinor field—in terms of pair currents $j^{\pm}{}_{\mu}(\xi)$, which are periodic and regular functions. The 'conservation laws'

$$\partial_{\mp} j^{\pm}{}_{\mu}(\xi^0, \xi^1) = 0$$

take place for these currents.

Due to conditions (29*a*), the correspondence

$$(X_\mu, \Psi_{\pm}{}^{a}) \leftrightarrow (j^{+}{}_{\mu}, j^{-}{}_{\mu})$$

is a one-to-one correspondence.

Equations (28*a*) define some constraints in the phase space of the string. As follows from the results of Talalov (1987), the consequence of (31) for the non-periodic case is

$$\{M_{\pm} \underset{,}{\otimes} Q_{\pm}(\xi)\} = 0.$$

That is why we do not expect any trouble from conditions (28*a*). Detailed discussions of similar problems can be found in Jorjadze *et al* (1986).

Due to periodicity $X_\mu(\xi^1)$ another constraint exists in our model. For zero modes $a^0{}_{\mu}$ of the Fourier decomposition of the functions $X'_\mu(\xi^1)$ we have

$$a^0_\mu \equiv \frac{1}{\pi} \int_0^\pi \mathrm{d}\xi^1((1/s)x_{+\mu} + sx_{-\mu}) = 0.$$

We put off a discussion of these conditions, which are important from the viewpoint of quantum theory, until future papers.

In terms of currents $j^{\pm}{}_{\mu}(\xi)$,

$$\theta_{\pm}(\xi) = -\tfrac{1}{4} j^{\pm}{}_{\mu}(\xi) j^{\pm\mu}(\xi)$$

and, as a direct consequence of (33), we have

$$\{\theta_{\pm}(\xi), \theta_{\pm}(\eta)\} = \mp[\theta_{\pm}(\xi) + \theta_{\pm}(\eta)]\delta'(\xi - \eta) \tag{34}$$

$$\{\theta_{+}(\xi), \theta_{-}(\eta)\} = 0.$$

We should stress that, in our approach, the current algebras (33) are the most fundamental objects, unlike Virasoro algebras (34), which are only the second most fundamental. The reason is that dynamical variables of the model are not only coefficients of the first quadratic form but also coefficients of the second quadratic form of the surface $\{X_\mu\}$.

5. Concluding remarks

Equation (10) for matrix $K \in \mathrm{SL}(2, R)$ is a particular case of the general equation

$$\partial_{+}(g^{-1}\partial_{-g}) + \varkappa\partial_{-}(g^{-1}\partial_{+g}) = 0 \tag{35}$$

for a two-dimensional chiral field g with an anomaly which was suggested by Novikov (1980). The linear representation for (35) was obtained by Volovich (1985), but our degenerate case for one system (20) can only be deduced from this general linear representation by means of the limiting procedure $\varkappa \to \infty$.

The appearance of a current algebra as the foundation of Poisson structure of the main chiral field ($\varkappa = 1$) is well known (Takhtajan and Faddeev 1986). Theories with $\varkappa = 0$ and $\varkappa = \infty$ are similar (Witten 1984).

Of course, our preliminary (classical) considerations of the suggested string model do not help to solve any global problems in string theory. However, the appearance of such fundamental objects as a current algebra with central charges permits us to hope for some interesting results in the quantum case.

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